Fourier transform

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February 5, 2021

The discrete Fourier transform (DFT) is a computational tool to work with signals that are defined on a discrete time support and contain a finite number of elements. Time in the world is neither discrete nor finite, which motivates consideration of continuous time signals $x : \mathbb{R} \to \mathbb{C}$. These signals map a continuous time index $t \in \mathbb{R}$ to a complex value $x(t) \in \mathbb{C}$. The signal values $x(t)$ can be, and often are, real.

Paralleling the development performed for discrete signals, we define the Fourier transform of the continuous time signal x as the signal $X : \mathbb{R} \to \mathbb{C}$ for which the signal values $X(f)$ are given by the integral

$$X(f) := \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt. \tag{1}$$

The definition in (1) is different in form to the definition of the DFT, but it is conceptually analogous. Whatever intuition we have gained so far on dealing with the DFT of discrete signals extends more or less unchanged to the Fourier transform of continuous signals.

The statement above has a very deep meaning that will become clear once we develop the theory of sampling. For the time being we can observe that the DFT can be considered as an approximation of the Fourier transform in which we start with N samples of x to obtain N samples of X. To see that this is true, consider N samples of x, separated by a sampling time T_s, and extending between times $t = 0$ and $t = NT_s$. The Riemann approximation of the integral in (1) is then given by

$$X(f) \;=\; \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \;\approx\; T_s \sum_{n=0}^{N-1} x(nT_s) e^{-j2\pi f n T_s}. \tag{2}$$

The approximation above is true for all frequencies, but if we just consider

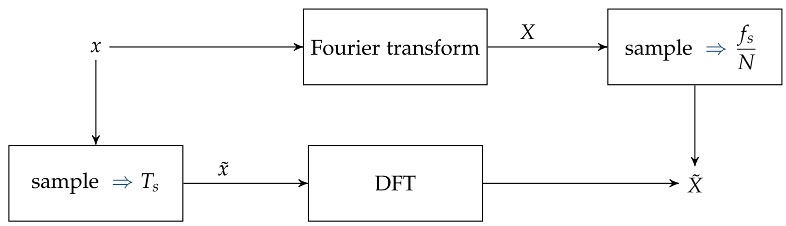

Figure 1. The discrete Fourier transform provides a numerical approximation to the Fourier transform.

the frequencies $f = (k/N)f_s$ for $k \in [-N/2, N/2]$ we can rewrite (2) as

$$X\left(\frac{k}{N}f_s\right) \approx T_s \sum_{n=0}^{N-1} x(nT_s)e^{-j2\pi(k/N)f_s nT_s} = T_s \sum_{n=0}^{N-1} x(nT_s)e^{-j2\pi kn/N}, \tag{3}$$

where we used the fact that $f_s = 1/T_s$. Except for constants, the rightmost side of (3) is the definition of the DFT of the discrete signal $\tilde{x}$ with components $\tilde{x}(n) = x(nT_s)$. Indeed, the DFT $\tilde{X} = \mathcal{F}(\tilde{x})$ of the discrete signal $\tilde{x}$ has components

$$\tilde{X}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \tilde{x}(n)e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_s)e^{-j2\pi kn/N}. \tag{4}$$

Upon comparison of (3) and (4) we can conclude that the DFT $\tilde{X}$ of the sampled signal $\tilde{x}$ and the Fourier transform X of the continuous signal x are approximately related by the expression

$$\tilde{X}(k) \approx \frac{1}{T_s\sqrt{N}} X\left(\frac{k}{N}f_s\right). \tag{5}$$

The relationship in (5) allows us to approximate the Fourier transform of a signal with numerical operations, or, conversely, to conclude that a property derived for Fourier transforms is approximately valid for DFTs as well. The approximating relationship in (5) is represented schematically in Figure 1. In this lab we will use (5) to verify numerically some formulas that we will derive analytically.

1 Computation of Fourier transforms

We define a Gaussian pulse of standard deviation σ and average value μ as the signal x with values $x(t)$ given by the formula

$$x(t) = e^{-(t-\mu)^2/(2\sigma^2)}. \tag{6}$$

The standard deviation σ controls the width of the pulse. Large σ corresponds to wide pulses and small σ corresponds to narrow pulses. The mean value μ controls the location of the pulse on the real line.

1.1 Fourier transform of a Gaussian pulse. Derive an expression for the Fourier transform of the Gaussian pulse when $\mu = 0$. You will have to make use of the fact that the integral

$$\int_{-\infty}^{\infty} x_\sigma(t) = \int_{-\infty}^{\infty} e^{-t^2/(2\sigma^2)} dt = \sqrt{2\pi\sigma^2}. \tag{7}$$

1.2 Numerical verification. Verify numerically that your derivation in Part 1.1 is correct. You will have to be careful with the selection of your sampling time and sampling interval. Try the comparison for different values of σ. Report for $\sigma = 1$, $\sigma = 2$, and $\sigma = 4$.

1.3 Fourier transform of a shifted Gaussian pulse. Derive an expression for the Fourier transform of the Gaussian pulse for generic μ. Verify numerically. The solution to this part is very easy once you have solved Part 1.1.

2 Modulation and demodulation

An important property of Fourier transforms is that shifting a signal in the time domain is equivalent to multiplying by a complex exponential in the frequency domain. More specifically consider a given signal x and shift τ and define shifted signal x_τ as

$$x_\tau = x(t - \tau) \tag{8}$$

The Fourier transform of x is denoted as $X = \mathcal{F}(x)$ and the Fourier transform of x_τ is denoted as $X_\tau = \mathcal{F}(x_\tau)$. We then have that the following theorem holds true.

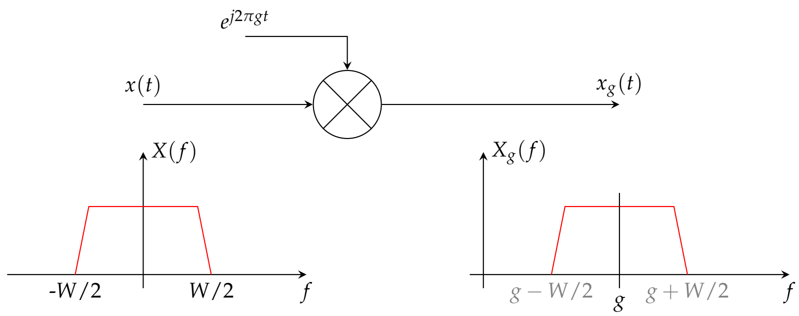

Figure 2. Modulation of a bandlimited signal. The bandlimited spectrum of signal x is re-centered at frequency g when the signal is multiplied by a complex exponential of frequency g.

Theorem 1 *A time shift of τ units in the time domain is equivalent to multiplication by a complex exponential of frequency $-\tau$ in the frequency domain*

$$x_\tau = x(t-\tau) \qquad \Longleftrightarrow \qquad X_\tau(f) = e^{-j2\pi f\tau}X(f) \tag{9}$$

These result has important applications, the most popular of which is its use in signal detection. This application utilizes the fact that the moduli of X and X_τ are the same, which allows the comparison of signals without worrying about the selection of the time origin.

A property that we can call the dual of the result in Theorem 1 is that multiplying a signal by a complex exponential results in a shift in the frequency domain. Specifically, for given signal x and frequency g, we define the modulated signal

$$x_g(t) = e^{j2\pi gt}x(t) \tag{10}$$

We write the Fourier transform of x as $X = \mathcal{F}(x)$ and the Fourier transform of x_g as $X_g = \mathcal{F}(x_g)$. We then have that the following theorem holds true.

Theorem 2 *A multiplication by a complex exponential of frequency g in the time domain is equivalent to a shift of g units in the frequency domain*

$$x_g(t) = e^{j2\pi gt}x(t) \qquad \Longleftrightarrow \qquad X_g(f) = X(f-g) \tag{11}$$

Proof: Write down a proof of Theorem 2. That is, prove that if $x_g(t) = e^{j2\pi gt}x(t)$ we must have $X_g(f) = X(f - g)$. ■

Despite looking less interesting than the claim in Theorem 1, the result in Theorem 2 is at least of equal importance because of its application in the modulation and demodulation of bandlimited signals. To explain this statement better, we begin with the definition of a bandlimited signal that we formally introduce next.

Definition 1 *The signal x with Fourier transform $X = \mathcal{F}(x)$ is said bandlimited with bandwidth W if we have $X(f) = 0$ for all frequencies $f \notin [-W/2, W/2]$.*

An illustration of the spectrum of a bandlimited signal is shown in Figure 2, where we also show the result of multiplying x by a complex exponential of frequency g. When we do that, the spectrum is re-centered at the modulating frequency g. Signals that are literally bandlimited are hard to find, but signals that are approximately bandlimited do exist. As an example, we consider voice recordings.

2.1 Voice as a bandlimited signal. Record 3 seconds of the voice of one of your group members at a sampling rate of 40kHz. Feel free to use the provided python class `recordsound()` to do this. Take the DFT of your voice and observe that coefficients with frequencies $f > 4$kHz are close to null. Set these coefficients to zero to create a bandlimited signal. Play your voice back and observe that the removed frequencies don't affect the quality of your voice.

2.2 Voice modulation. Take the bandlimited signal you created in Part 2.1 and modulate it with center frequency $g_1 = 5$kHz.

2.3 Modulation with a cosine. The problem with modulating with a complex exponential as we did in Part 2.2 is that complex exponentials are signals with imaginary parts that, therefore, can't be generated in a real system. In a real system we have to modulate using a cosine or a sine. Redefine then the modulated signal as

$$x_g(t) = cos(2\pi gt)x(t), \tag{12}$$

and let $X_g = \mathcal{F}(x_g)$ be the respective Fourier transform. Write down an expression for X_g in terms of X (it might be useful to recall how to write the $\cos(x)$ function as a combination of complex exponentials). Take the bandlimited signal you created in Part 2.1 and modulate it with a cosine with frequency $g_1 = 5$kHz. Verify that expression you derived is correct.

2.4 Another voice signal. Record the voice of another group member and repeat Part 2.1. Repeat Part 2.3 but use a cosine with frequency $g_2 = 13\text{kHz}$. Sum up the respective modulated signals to create the mixed signal z.

2.5 Recover individual voices. Explain how to recover your voice and the voice of your partner from the mixed signal z. Implement the recovery and play back the individual voice pieces.

3 Time management

This lab is designed to be a respite from the more intensive Lab 3. Part 1 should take 2 hours and Part 2 between 4 and 5. To solve Part 1 you need to make use of a technique called "completing squares." If you have never done that, ask one of your teaching assistants right away. To solve Part 2 do remember to make use of our help. There's no reason to struggle when you can receive help.