Discrete Fourier transform

Alejandro Ribeiro
Dept. of Electrical and Systems Engineering
University of Pennsylvania
aribeiro@seas.upenn.edu
http://www.seas.upenn.edu/users/~aribeiro/

January 21, 2022

- Discrete complex exponential of discrete frequency k and duration N

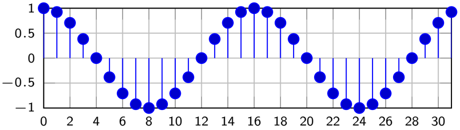
$$e_{kN}(n) = \frac{1}{\sqrt{N}}\, e^{j2\pi kn/N} = \frac{1}{\sqrt{N}}\, \exp(j2\pi kn/N)$$

- The complex exponential is explicitly given by

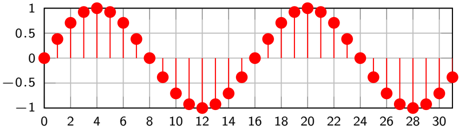
$$e^{j2\pi kn/N} = \cos(2\pi kn/N) + j\sin(2\pi kn/N)$$

- Real part is a discrete cosine and imaginary part a discrete sine

$\text{Re}\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



$\text{Im}\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



Equivalent frequencies

Theorem

If $k - l = N$ the signals $e_{kN}(n)$ and $e_{lN}(n)$ coincide for all n, i.e.,

$$e_{kN}(n) = \frac{e^{j2\pi kn/N}}{\sqrt{N}} = \frac{e^{j2\pi ln/N}}{\sqrt{N}} = e_{lN}(n)$$

- Although there are infinite possible frequencies complex exponentials with frequencies k and l are equivalent when the difference $k - l = N$ (or $k - l = \dot{N}$)
- Only frequencies between 0 and $N - 1$ are meaningful. Or, only frequencies between $-N/2 + 1$ and $N/2$ are meaningful.

Theorem

Opposite frequencies k and $-k$ yield conjugate signals: $e_{-kN} = e^*_{kN}(n)$

Proof.

- Just use the definitions to write the chain of equalities

$$e_{-kN}(n) = \frac{e^{j2\pi(-k)n/N}}{\sqrt{N}} = \frac{e^{-j2\pi kn/N}}{\sqrt{N}} = \left[\frac{e^{j2\pi kn/N}}{\sqrt{N}}\right]^* = e^*_{kN}(n) \qquad \square$$

- Opposite frequencies have the same real part and opposite imaginary part. The cosine is the same, the sine changes sign
- Only frequencies between 0 and $N/2$ are meaningful. This is fitting, as we can't have an oscillation with more than $N/2$ periods

Theorem

Complex exponentials with nonequivalent frequencies are orthogonal. I.e.

$$\langle e_{kN}, e_{lN} \rangle = 0$$

when $k - l < N$. *E.g., when* $k = 0, \ldots N-1$, *or* $k = -N/2+1, \ldots, N/2$.

- Signals of canonical sets are "unrelated." Different rates of change
- Also note that the energy is $\|e_{kN}\|^2 = \langle e_{kN}, e_{kN} \rangle = 1$
- Exponentials with frequencies $k = 0, 1, \ldots, N-1$ are orthonormal

$$\langle e_{kN}, e_{lN} \rangle = \delta(l - k)$$

- They are an orthonormal basis of signal space with N samples

Proof.

- Use definitions of inner product and discrete complex exponential to write

$$\langle e_{kN}, e_{lN} \rangle = \sum_{n=0}^{N-1} e_{kN}(n) e^*_{lN}(n) = \sum_{n=0}^{N-1} \frac{e^{j2\pi kn/N}}{\sqrt{N}} \frac{e^{-j2\pi ln/N}}{\sqrt{N}}$$

- Regroup terms to write as geometric series

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} \left[e^{j2\pi(k-l)/N} \right]^n$$

- Geometric series with basis a sums to $\sum_{n=0}^{N-1} a^n = (1-a^N)/(1-a)$. Thus,

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \frac{1 - \left[e^{j2\pi(k-l)/N} \right]^N}{1 - e^{j2\pi(k-l)/N}} = \frac{1}{N} \frac{1 - 1}{1 - e^{j2\pi(k-l)/N}} = 0$$

- Completed proof by noting $\left[e^{j2\pi(k-l)/N} \right]^N = e^{j2\pi(k-l)} = \left[e^{j2\pi} \right]^{(k-l)} = 1$ $\square$

Definition of discrete Fourier transform (DFT)

- Signal x of duration N with elements $x(n)$ for $n = 0, \ldots, N-1$
- X is the discrete Fourier transform (DFT) of x if for all $k \in \mathbb{Z}$

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn/N)$$

- We write $X = \mathcal{F}(x)$. All values of X depend on all values of x
- The argument k of the DFT is referred to as frequency
- DFT is complex even if signal is real $\Rightarrow X(k) = X_R(k) + jX_I(k)$

$\Rightarrow$ It is customary to focus on magnitude

$$\left|X(k)\right| = \left[X_R^2(k) + X_I^2(k)\right]^{1/2} = \left[X(k)X^*(k)\right]^{1/2}$$

DFT elements as inner products

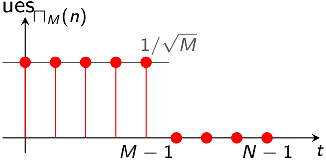
- Discrete complex exponential (freq. k) $\Rightarrow e_{-kN}(n) = \frac{1}{\sqrt{N}} e^{-j2\pi kn/N}$
- Can rewrite DFT as $\Rightarrow X(k) = \sum_{n=0}^{N-1} x(n) e_{-kN}(n) = \sum_{n=0}^{N-1} x(n) e_{kN}^{*}(n)$
- And from the definition of inner product $\Rightarrow X(k) = \langle x, e_{kN} \rangle$
- DFT element $X(k)$ $\Rightarrow$ inner product of $x(n)$ with $e_{kN}(n)$

  $\Rightarrow$ Projection of $x(n)$ onto complex exponential of frequency k

  $\Rightarrow$ How much of the signal x is an oscillation of frequency k

- The unit energy square pulse is the signal $\sqcap_M(n)$ that takes values

$$\sqcap_M(n) = \frac{1}{\sqrt{M}} \qquad \text{if } 0 \leq n < M$$
$$\sqcap_M(n) = 0 \qquad \text{if } M \leq n$$



- Since only the first $M-1$ elements of $\sqcap_M(n)$ are not null, the DFT is

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sqcap_M(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{n=0}^{M-1} \frac{1}{\sqrt{M}} e^{-j2\pi kn/N}$$

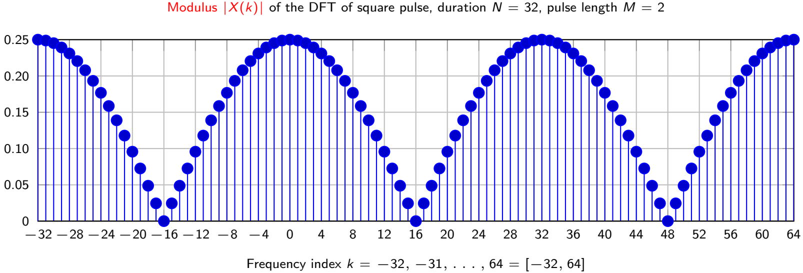
- $X(k) =$ sum of first M components of exponential of frequency $-k$
- Can reduce to simpler expression but who cares? $\Rightarrow$ It's just a sum

DFT of a square pulse (illustration)

- Square pulse of length $M = 2$ and overall signal duration $N = 32$

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{1} \frac{1}{\sqrt{2}} e^{-j2\pi kn/N} = \frac{1}{\sqrt{2N}} \left(1 + e^{-j2\pi k/N}\right)$$

- E.g., $X(k) = \dfrac{2}{\sqrt{2N}}$ at $k = 0, \pm N, \ldots$ and $X(k) = 0$ at $k = 0 \pm N/2, \pm 3N/2, \ldots$



- This DFT is periodic with period N $\Rightarrow$ true in general

- Consider frequencies k and $k+N$. The DFT at $k+N$ is

$$X(k+N) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N}$$

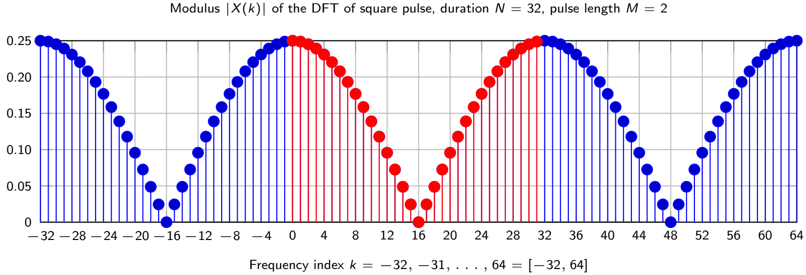
- Complex exponentials of freqs. k and $k+N$ are equivalent. Then

$$X(k+N) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = X(k)$$

- DFT values N apart are equivalent $\Rightarrow$ DFT has period N
- Suffices to look at N consecutive frequencies $\Rightarrow$ canonical sets
  - $\Rightarrow$ Computation $\Rightarrow k \in [0, N-1]$
  - $\Rightarrow$ Interpretation $\Rightarrow k \in [-N/2, N/2]$ (actually, $N+1$ freqs.)
  - $\Rightarrow$ Related by chop and shift $\Rightarrow [-N/2, -1] \sim [N/2, N-1]$

Canonical set $k \in [0, N-1]$

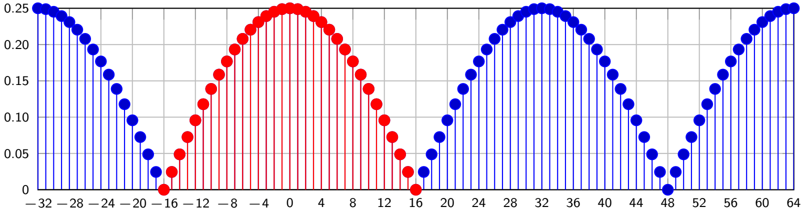
- DFT of the square pulse highlighting frequencies $k \in [0, N-1]$



- Frequencies larger than $N/2$ have no clear physical meaning

- DFT of the square pulse highlighting frequencies $k \in [-N/2, N/2]$
- Negative freq. $-k$ has the same interpretation as positive freq. k
- One redundant element $\Rightarrow X(-N/2) = X(N/2)$. Just convenient

Modulus $|X(k)|$ of the DFT of square pulse, duration $N = 32$, pulse length $M = 2$



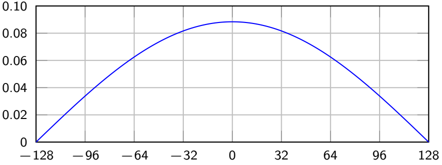
Frequency index $k = -32, -31, \ldots, 64 = [-32, 64]$

- Obtain frequencies $k \in [-N/2, -1]$ from frequencies $[N/2, N-1]$

Pulses of different length

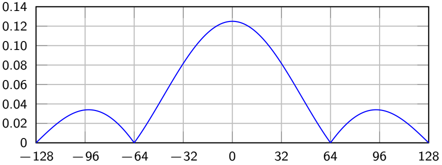
- The DFT X gives information on how fast the signal x changes

DFT modulus of square pulse, duration $N = 256$, pulse length $M = 2$



Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

DFT modulus of square pulse, duration $N = 256$, pulse length $M = 4$

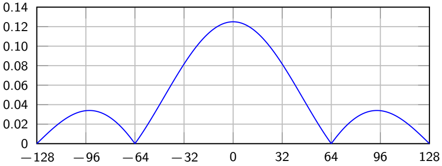


Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

- For length $M = 2$ have weight at high frequencies
- Length $M = 4$ concentrates weight at lower frequencies
- Pulse of length $M = 2$ changes more than a pulse of length $M = 4$

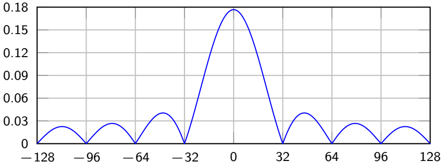
- The lengthier the pulse the less it changes $\Rightarrow$ DFT concentrates at zero freq.

DFT modulus of square pulse, duration $N = 256$, pulse length $M = 4$



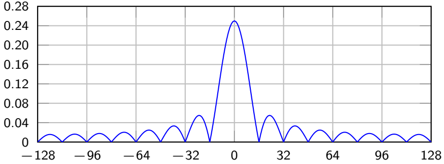
Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

DFT modules of square pulse, duration $N = 256$, pulse length $M = 8$



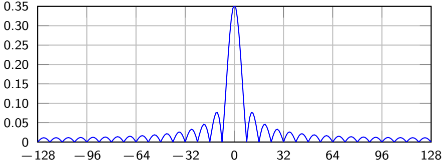
Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

DFT modulus of square pulse, duration $N = 256$, pulse length $M = 16$



Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

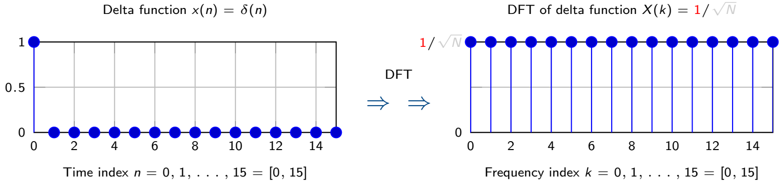
DFT modulus of square pulse, duration $N = 256$, pulse length $M = 32$



Frequency index $k = -128, -127, \ldots, 128 = [-128, 128]$

- The delta function is $\delta(0) = 1$ and $\delta(n) = 0$, else. Then, the DFT is

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \delta(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \delta(0) e^{-j2\pi k0/N} = \frac{1}{\sqrt{N}}$$



- Only the N values $k \in [0, 15]$ shown. DFT defined for all k but periodic
- Observe that the energy is conserved $\|X\|^2 = \|\delta\|^2 = 1$

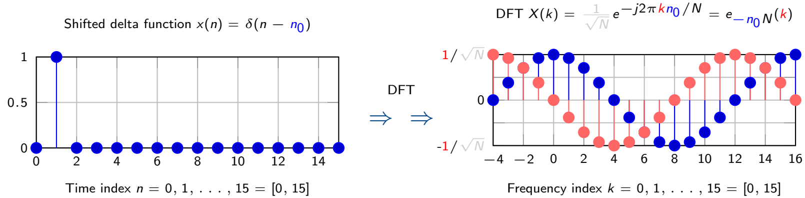
- For shifted delta $\delta(n_0 - n_0) = 1$ and $\delta(n - n_0) = 0$ otherwise. Thus

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \delta(n - n_0) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \delta(n_0 - n_0) e^{-j2\pi kn_0/N}$$

- Of course $\delta(n_0 - n_0) = \delta(0) = 1$, implying that

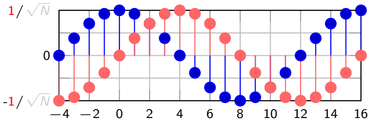
$$X(k) = \frac{1}{\sqrt{N}} e^{-j2\pi kn_0/N} = e_{-n_0N}(k)$$

- Complex exponential of frequency $-n_0$ (below, $N = 16$ and $n_0 = 1$)



- Complex exponential of freq. k_0 $\Rightarrow x(n) = \frac{1}{\sqrt{N}} e^{j2\pi k_0 n/N} = e_{k_0 N}(n)$
- Use inner product form of DFT definition $\Rightarrow X(k) = \langle e_{k_0 N}, e_{kN} \rangle$
- Orthonormality of complex exponentials $\Rightarrow \langle e_{k_0 N}, e_{kN} \rangle = \delta(k - k_0)$

Complex exponential $x(n) = \frac{1}{\sqrt{N}} e^{j2\pi k_0 n/N} = e_{k_0 N}(n)$

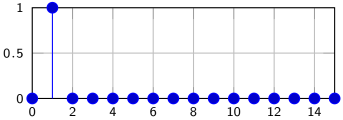


Time index $n = 0, 1, \ldots, 15 = [0, 15]$

DFT

$\Rightarrow \Rightarrow$

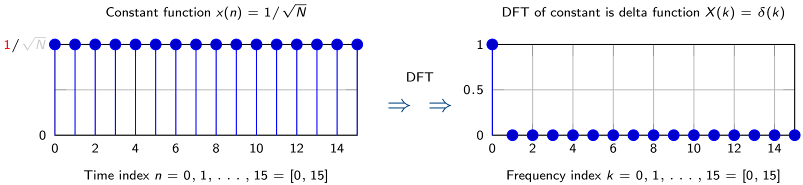
DFT is shifted delta function $X(k) = \delta(k - k_0)$



Frequency index $n = 0, 1, \ldots, 15 = [0, 15]$

- DFT of exponential $e_{k_0 N}(n)$ is shifted delta $X(k) = \delta(k - k_0)$

- Constant function $x(n) = 1/\sqrt{N}$ (it has unit energy) and $k = 0$
  $\Rightarrow$ Complex exponential with frequency $k_0 = 0 \;\Rightarrow x(n) = e_{0N}$
- Use inner product form of DFT definition $\;\Rightarrow X(k) = \langle e_{0N}, e_{kN} \rangle$
- Complex exponential orthonormality $\;\Rightarrow \langle e_{0N}, e_{kN} \rangle = \delta(k - 0) = \delta(k)$



- DFT of constant $x(n) = 1/\sqrt{N}$ is delta function $X(k) = \delta(k)$

- DFT of a signal captures its rate of change
- Signals that change faster have more DFT weight at high frequencies

- DFT conserves energy (all have unit energy in our examples)
- Energy of DFT $X = \mathcal{F}(x)$ is the same as energy of the signal x
- Indeed, an important property we will show

- Duality of signal - transform pairs (signals and DFTs come in pairs)
- DFT of delta is a constant. DFT of constant is a delta
- DFT of exponential is shifted delta. DFT of shifted delta is exponential
- Indeed, a fact that follows from the form of the inverse DFT

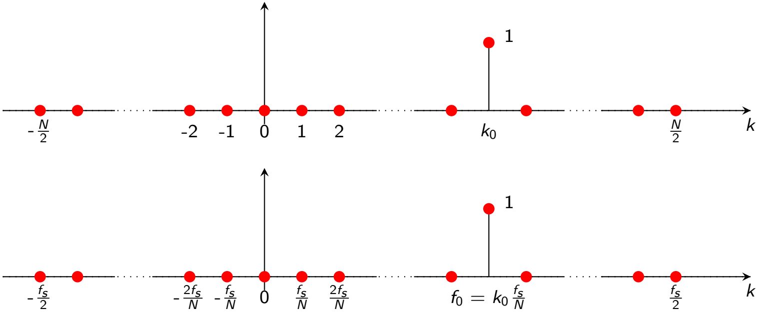
Units

- Sampling time T_s, sampling frequency f_s, signal duration $T = NT_s$
- Discrete frequency $k \Rightarrow k$ oscillations in time NT_s = Period NT_s/k
- Discrete frequency k equivalent to real frequency $f_k = \dfrac{k}{NT_s} = k\dfrac{f_s}{N}$
- In particular, $k = N/2$ equivalent to $\Rightarrow f_{N/2} = \dfrac{N/2 f_s}{N} = \dfrac{f_s}{2}$
- Set of frequencies $k \in [-N/2, N/2]$ equivalent to real frequencies ...
  - $\Rightarrow$ That lie between $-f_s/2$ and $f_s/2$
  - $\Rightarrow$ Are spaced by f_s/N (difference between frequencies f_k and f_{k+1})
- Interval width given by sampling frequency. Resolution given by N

- Complex exponential of frequency $f_0 = k_0 f_s / N$

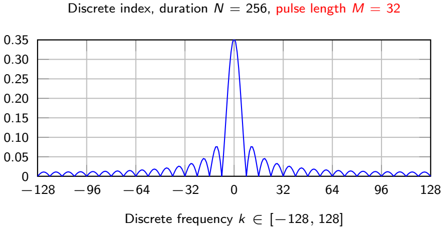
  $\Rightarrow$ Discrete frequency k_0 and DFT $\Rightarrow X(k) = \delta(k - k_0)$

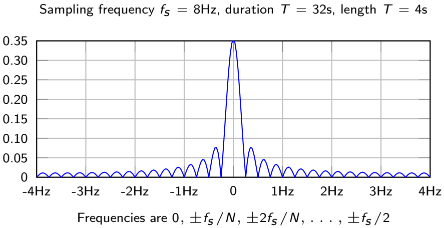
- But frequency k_0 corresponds to frequency f_0 $\Rightarrow X(f) = \delta(f - f_0)$



- True only when frequency $f_0 = (k_0/N) f_s$ is a multiple of f_s/N

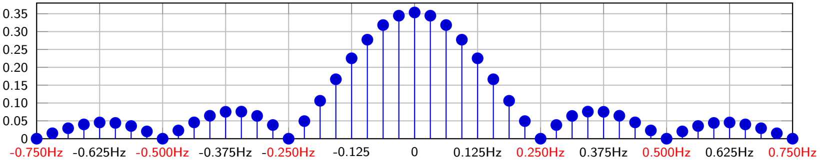
- Square pulse of length $T_0 = 4$s observed during a total of $T = 32$s.
- Sampled every $T_s = 125$ms $\Rightarrow$ Sample frequency $f_s = 8$Hz
- Total number of samples $\Rightarrow N = T/T_s = 256$
- Maximum frequency $k = N/2 = 128 \leftrightarrow f_k = f_{N/2} = f_s/2 = 4Hz$
- Fequency resolution $f_s/N = 8\text{Hz}/256 = 0.03125\text{Hz}$





- Interval between freqs. $\Rightarrow f_s/N = 8\text{Hz}/256 = 1/32 = 0.03125\text{Hz}$
  $\Rightarrow$ 32 equally spaced frees for each 1Hz interval = 8 every 0.125 Hz.

Sampling frequency $f_s = 8\text{Hz}$, duration $T = 32\text{s}$, length $T = 4\text{s}$



Frequencies $0, \pm 0.03125Hz, \pm 0.06250Hz, \ldots, \pm 0.750Hz$

- Zeros of DFT are at frequencies 0.250Hz, 0.500 Hz, 0.750 Hz, . . .
  $\Rightarrow$ Thus, zeros are at frequencies are $1/T_0, 2/T_0, 3/T_0, \ldots$
- Most (a lot) of the DFT energy is between freqs. $-1/T_0$ and $1/T_0$

- Given a Fourier transform X, the inverse (i)DFT $x = \mathcal{F}^{-1}(X)$ is

$$x(n) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) \exp(j2\pi kn/N)$$

- Same as DFT but for sign in the exponent (also, sum over k, not n)

- Any summation over N consecutive frequencies works as well. E.g.,

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$$

- Because for a DFT X we know that it must be $X(k+N) = X(k)$

Theorem

The inverse DFT of the DFT of x is the signal x $\Rightarrow \mathcal{F}^{-1}[\mathcal{F}(x)] = x$

- Every signal x can be written as a sum of complex exponentials

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$$

- Coefficient multiplying $e^{j2\pi kn/N}$ is $X(k) = k$th element of DFT of x

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Proof.

- Let $X = \mathcal{F}(x)$ be the DFT of x. Let $\tilde{x} = \mathcal{F}^{-1}(X)$ be the iDFT of X.

  $\Rightarrow$ We want to show that $\tilde{x} \equiv x$

- From the definition of the iDFT of $X \Rightarrow \tilde{x}(\tilde{n}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k) e^{j2\pi k\tilde{n}/N}$

- From the definition of the DFT of $x \Rightarrow X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

- Substituting expression for $X(k)$ into expression for $\tilde{x}(\tilde{n})$ yields

$$\tilde{x}(\tilde{n}) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right] e^{j2\pi k\tilde{n}/N}$$

Proof.

- Exchange summation order to sum first over k and then over n

$$\tilde{x}(\tilde{n}) = \sum_{n=0}^{N-1} x(n) \left[\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{j2\pi k\tilde{n}/N} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} \right]$$

- Pulled $x(n)$ out because it doesn't depend on k

- Innermost sum is the inner product between $e_{\tilde{n}N}$ and e_{nN}. Orthonormality:

$$\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{j2\pi k\tilde{n}/N} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} = \delta(\tilde{n} - n)$$

- Reducing to $\Rightarrow \tilde{x}(\tilde{n}) = \sum_{n=0}^{N-1} x(n)\delta(\tilde{n} - n) = x(\tilde{n})$

- Last equation is true because only term $n = \tilde{n}$ is not null in the sum $\square$

- Discrete complex exponential (freq. n) $\Rightarrow e_{nN}(k) = \frac{1}{\sqrt{N}} e^{j2\pi kn/N}$

- Rewrite iDFT as $\Rightarrow x(n) = \sum_{k=0}^{N-1} X(k) e_{nN}(k) = \sum_{k=0}^{N-1} X(k) e^{*}_{-nN}(k)$

- And from the definition of inner product $\Rightarrow x(n) = \langle X, e_{-nN} \rangle$

- iDFT element $X(k)$ $\Rightarrow$ inner product of $X(k)$ with $e_{-nN}(k)$

- Different from DFT, this is not the most useful interpretation

- Signal as sum of exponentials $\Rightarrow x(n) = \frac{1}{\sqrt{N}} \sum_{k=-N/2+1}^{N/2} X(k) e^{j2\pi kn/N}$

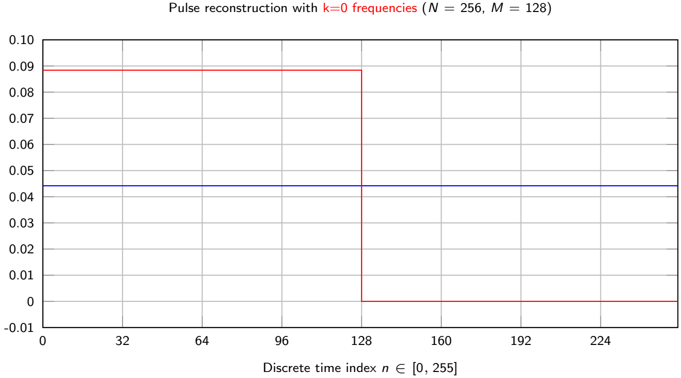
- Expand the sum inside out from $k=0$ to $k=\pm 1$, to $k=\pm 2$, ...

$$
\begin{array}{llllll}
x(n) = & X(0) & e^{j2\pi 0n/N} & & & \text{constant} \\
& + X(1) & e^{j2\pi 1n/N} & + X(-1) & e^{-j2\pi 1n/N} & \text{single oscillation} \\
& + X(2) & e^{j2\pi 2n/N} & + X(-2) & e^{-j2\pi 2n/N} & \text{double oscillation} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
& + X\left(\frac{N}{2}-1\right) & e^{j2\pi\left(\frac{N}{2}-1\right)n/N} & + X\left(-\frac{N}{2}+1\right) & e^{-j2\pi\left(\frac{N}{2}-1\right)n/N} & \left(\frac{N}{2}-1\right) - \text{oscillation} \\
& + X\left(\frac{N}{2}\right) & e^{j2\pi\left(\frac{N}{2}\right)n/N} & & & \frac{N}{2} - \text{oscillation}
\end{array}
$$

- Start with slow variations and progress on to add faster variations

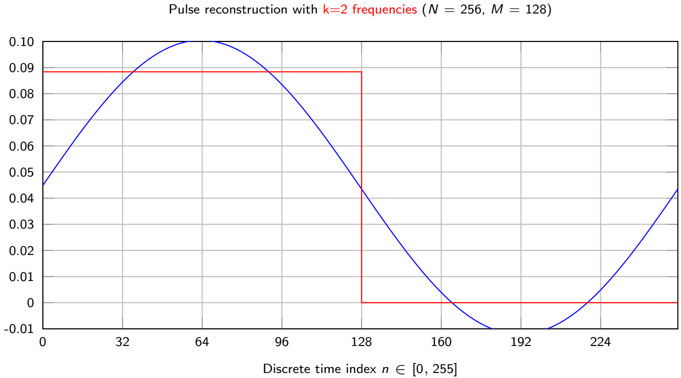
Reconstruction of square pulse

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequency $k = 0$ only (DC component)



- Bound to be not very good $\Rightarrow$ Just the average signal value

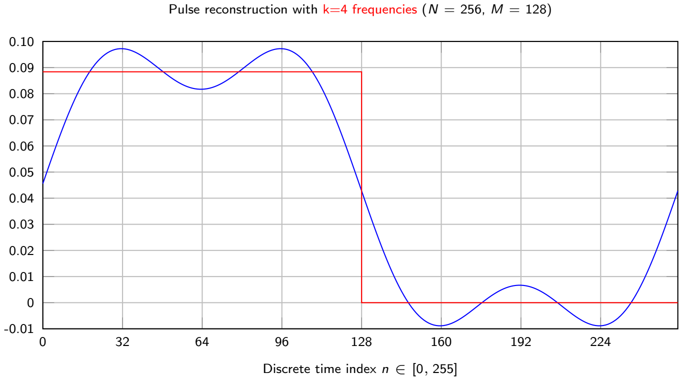
- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies $k = 0$, $k = \pm 1$, and $k = \pm 2$



- Not too bad, sort of looks like a pulse $\Rightarrow$ only 3 frequencies

Reconstruction of square pulse

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 4$

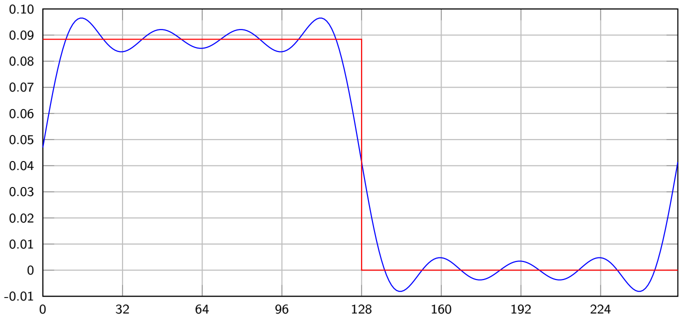


- Starts to look like a good approximation

Reconstruction of square pulse

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 8$

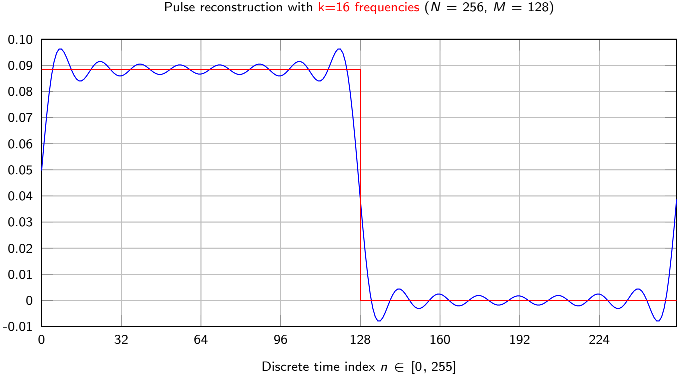
Pulse reconstruction with k=8 frequencies ($N = 256$, $M = 128$)



Discrete time index $n \in [0, 255]$

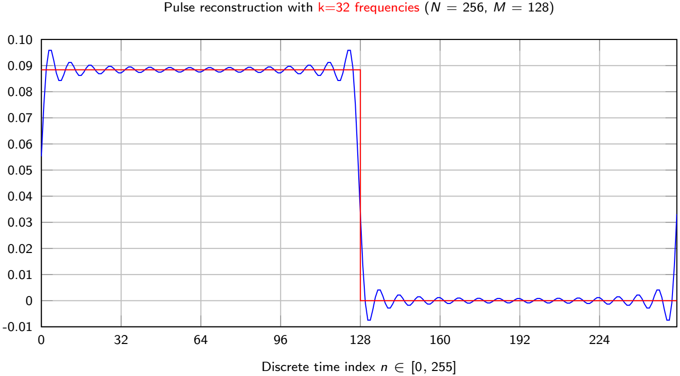
- Good approximation of the $N = 256$ values with 9 DFT coefficients

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 16$



- Compression $\Rightarrow$ Store $k + 1 = 17$ DFT values instead of $N = 128$ samples

- Consider square pulse of duration $N = 256$ and length $M = 128$
- Reconstruct with frequencies up to $k = 32$



- Can tradeoff less compression for better signal accuracy

Spectrum (re)shaping

(1) Start with a signal x with elements $x(n)$. Compute DFT X as

$$X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

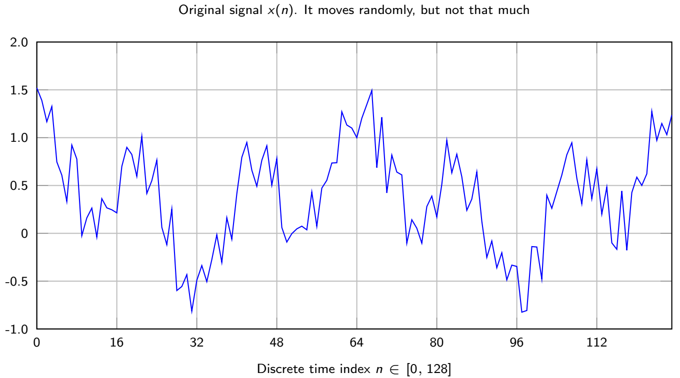
(2) (Re)shape spectrum $\Rightarrow$ Transform DFT X into DFT Y

(3) With DFT Y available, recover signal y with inverse DFT

$$y(n) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} Y(k) e^{j2\pi kn/N}$$

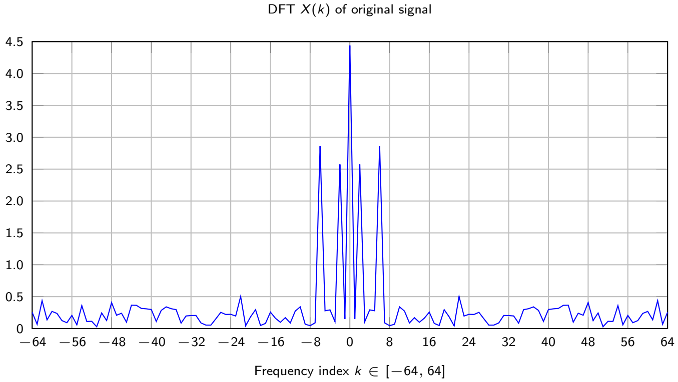
$$\xrightarrow{x} \boxed{\mathcal{F}} \xrightarrow{X} \boxed{\text{SS}} \xrightarrow{Y} \boxed{\mathcal{F}^{-1}} \xrightarrow{y}$$

- An application of spectrum reshaping is to clean a noisy signal
- Signal with some underlying trend (good) and some noise (bad)



- Which is which? $\Rightarrow$ Not clear $\Rightarrow$ Let's look at the spectrum (DFT)

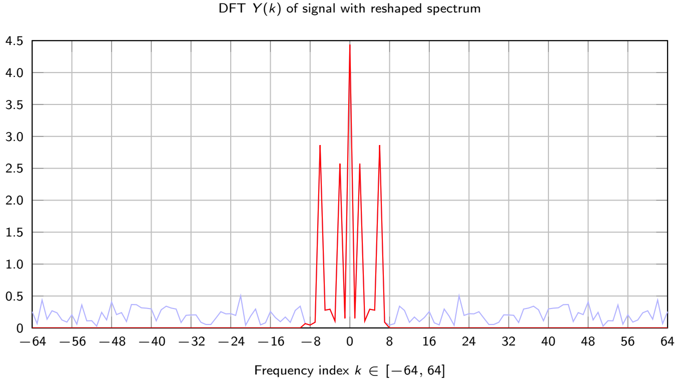
- An application of spectrum reshaping is to clean a noisy signal
- Now the trend (spikes) is clearly separated from the noise (the floor)



- How do we remove the noise? $\Rightarrow$ Reshape the spectrum

Spectrum reshaping to remove noise

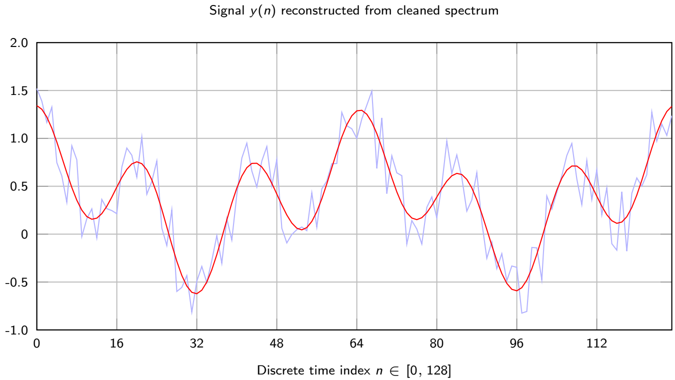
- An application of spectrum reshaping is to clean a noisy signal
- Remove freqs. larger than 8 $\Rightarrow Y(k) = 0$ for $k > 8$, $Y(k) = X(k)$ else



- How do we recover the trend? $\Rightarrow$ Inverse DFT

Spectrum reshaping to remove noise

- An application of spectrum reshaping is to clean a noisy signal
- Inverse DFT of reshaped specturm $Y(k)$ yields cleaned signal $y(n)$



- The trend now is clearly visible. Noise has been removed

Three important properties of DFTs

- DFTs of real signals (no imaginary part) are conjugate symmetric

$$X(-k) = X^*(k)$$

- Signals of unit energy have transforms of unit energy
- More generically, the DFT preserves energy (Parseval's theorem)

$$\sum_{n=0}^{N-1} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \sum_{k=0}^{N-1} |X(k)|^2$$

- The DFT operator is a linear operator

$$\mathcal{F}(ax + by) = a\mathcal{F}(x) + b\mathcal{F}(y)$$

Theorem

The DFT $X = \mathcal{F}(x)$ of a real signal x is conjugate symmetric

$$X(-k) = X^*(k)$$

- Can recover all DFT components from those with freqs. $k \in [0, N/2]$
- What about components with freqs. $k \in [-N/2, -1]$?

  $\Rightarrow$ Conjugates of those with freqs $k \in [0, N/2]$
- Other elements are equivalent to one in $[-N/2, N/2]$ (periodicity)

Proof of symmetry property

Proof.

- Write the DFT $X(-k)$ using its definition

$$X(-k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(-k)n/N}$$

- When the signal is real, its conjugate is itself $\Rightarrow x(n) = x^*(n)$
- Conjugating a complex exponential $\Rightarrow$ changing the exponent's sign

- Can then rewrite $\Rightarrow X(-k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x^*(n) \left(e^{-j2\pi kn/N}\right)^*$

- Sum and multiplication can change order with conjugation

$$X(-k) = \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}\right]^* = X^*(k) \qquad \square$$

Theorem (Parseval)

Let $X = \mathcal{F}(x)$ be the DFT of signal x. The energies of x and X are the same, i.e.,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \|x\|^2 = \|X\|^2 = \sum_{k=0}^{N-1} |X(k)|^2$$

- In energy of DFT, any set of consecutive freqs. would do. E.g.,

$$\|X\|^2 = \sum_{k=0}^{N-1} |X(k)|^2 = \sum_{k=-N/2+1}^{N/2} |X(k)|^2$$

Proof.

- From the definition of the energy of $X \Rightarrow \|X\|^2 = \sum_{k=0}^{N-1} X(k)X^*(k)$
- From the definition of the DFT of $x \Rightarrow X(k) := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$
- Substitute expression for $X(k)$ into one for $\|X\|^2$ (observe conjugation)

$$\|X\|^2 = \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}\right]\left[\frac{1}{\sqrt{N}} \sum_{\tilde{n}=0}^{N-1} x^*(\tilde{n})e^{+j2\pi k\tilde{n}/N}\right]$$

Proof.

- Distribute product and exchange order of summations $\Rightarrow$ sum over k first

$$\|X\|^2 = \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{N-1} x(n)x^*(\tilde{n}) \left[\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} \frac{1}{\sqrt{N}} e^{+j2\pi k\tilde{n}/N} \right]$$

- Pulled $x(n)$ and $x^*(\tilde{n})$ out because they don't depend on k

- Innermost sum is the inner product between $e_{\tilde{n}N}$ and e_{nN}. Orthonormality:

$$\sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{-j2\pi kn/N} \frac{1}{\sqrt{N}} e^{+j2\pi k\tilde{n}/N} = \langle e_{\tilde{n}N}, e_{nN} \rangle = \delta(\tilde{n} - n)$$

- Thus $\Rightarrow \|X\|^2 = \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{N-1} x(n)x^*(\tilde{n})\delta(\tilde{n} - n) = \sum_{n=0}^{N-1} x(n)x^*(n) = \|x\|^2$

- True because only terms $n = \tilde{n}$ are not null in the sum $\square$

Theorem

The DFT of a linear combination of signals is the linear combination of the respective DFTs of the individual signals,

$$\mathcal{F}(ax + by) = a\mathcal{F}(x) + b\mathcal{F}(y).$$

- In particular...
  - $\Rightarrow$ Adding signals ($z = x + y$) $\Rightarrow$ Adding DFTs ($Z = X + Y$)
  - $\Rightarrow$ Scaling signals($y = ax$) $\Rightarrow$ Scaling DFTs ($Y = aX$)

Proof.

- Let $Z := \mathcal{F}(ax + by)$. From the definition of the DFT we have

$$Z(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \Big[a x(n) + b y(n) \Big] e^{-j2\pi kn/N}$$

- Expand the product, reorder terms, identify the DFTs of x and y

$$Z(k) = \frac{a}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} + \frac{b}{\sqrt{N}} \sum_{n=0}^{N-1} y(n) e^{-j2\pi kn/N}$$

- First sum is DFT $X = \mathcal{F}(x)$. Second sum is DFT $Y = \mathcal{F}(y)$

$$Z(k) = aX(k) + bY(k)$$

□

DFT of a discrete cosine

- DFT of discrete cosine of freq. k_0 $\Rightarrow x(n) = \frac{1}{\sqrt{N}} \cos(2\pi k_0 n/N)$
- Can write cosine as a sum of discrete complex exponentials

$$x(n) = \frac{1}{2\sqrt{N}} \left[e^{j2\pi k_0 n/N} + e^{-j2\pi k_0 n/N} \right] = \frac{1}{2} \left[e_{k_0 N}(n) + e_{-k_0 N}(n) \right]$$

- From linearity of DFTs $\Rightarrow X = \mathcal{F}(x) = \frac{1}{2} \left[\mathcal{F}(e_{k_0 N}) + \mathcal{F}(e_{-k_0 N}) \right]$
- DFT of complex exponential e_{kN} is delta function $\delta(k - k_0)$. Then

$$X(k) = \frac{1}{2} \left[\delta(k - k_0) + \delta(k + k_0) \right]$$

- A pair of deltas at positive and negative frequency k_0

DFT of a discrete sine

- DFT of discrete sine of freq. k_0 $\Rightarrow x(n) = \frac{1}{\sqrt{N}} \sin(2\pi k_0 n/N)$
- Can write sine as a difference of discrete complex exponentials

$$x(n) = \frac{1}{2j\sqrt{N}} \left[e^{j2\pi k_0 n/N} - e^{-j2\pi k_0 n/N} \right] = \frac{-j}{2} \left[e_{k_0 N}(n) - e_{-k_0 N}(n) \right]$$

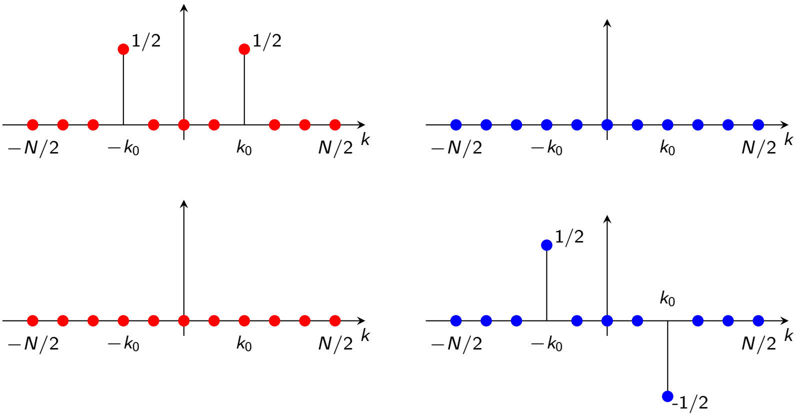
- From linearity of DFTs $\Rightarrow X = \mathcal{F}(x) = \frac{j}{2} \left[\mathcal{F}(e_{-k_0 N}) - \mathcal{F}(e_{k_0 N}) \right]$
- DFT of complex exponential e_{kN} is delta function $\delta(k - k_0)$. Then

$$X(k) = \frac{j}{2} \left[\delta(k + k_0) - \delta(k - k_0) \right]$$

- Pair of opposite complex deltas at positive and negative frequency k_0

DFT of discrete cosine and discrete sine

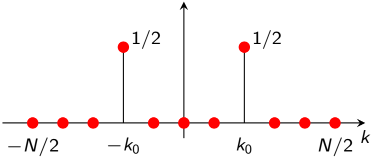
- Cosine has real part only (top). Sine has imaginary part only (bottom)

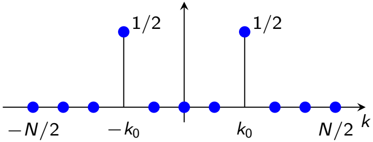


- Cosine is symmetric around $k = 0$. Sine is antisymmetric around $k = 0$.

DFT of discrete cosine and discrete sine (more)

- Real and imaginary parts are different but the moduli are the same





- Cosine and sine are essentially the same signal (shifted versions)
  - $\Rightarrow$ The moduli of their DFTs are identical
  - $\Rightarrow$ Phase difference captured by phase of complex number $X(\pm k_0)$